Mathematics Competition

Indiana University of Pennsylvania

2017

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not necessarily expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. The slope of the line between the points $(-2, -5)$ and $(5, -9)$ is:

   A. $-\frac{4}{7}$

   B. $-\frac{7}{4}$

   C. $\frac{4}{7}$

   D. $\frac{7}{4}$

   E. $-\frac{14}{3}$

---

2. When we simplify the expression $\sqrt{162x^9y^4z^5}$ the result is:

   A. $9x^4y^3z^2\sqrt{yz}$

   B. $9x^3y^2z^2\sqrt{2z}$

   C. $9x^4y^2z^2\sqrt{2xz}$

   D. $81x^4y^2z^2\sqrt{xz}$

   E. $81x^3y^2z^2\sqrt{2z}$

---

3. Danae packed 3 pairs of pants, 2 shirts, and 2 hats for a trip. If she always wears an outfit consisting of one pair of pants, one shirt, and one hat, then the number of different outfits she can make is:

   A. 7

   B. 6

   C. 12

   D. 81

   E. 24

---

4. The value of $\sin(240°)$ is:

   A. $-\frac{\sqrt{2}}{2}$

   B. 0

   C. $-\sqrt{3}$

   D. $-\frac{1}{2}$

   E. None of these

---

5. When $9^x + 9^x + 9^x$ is simplified to a single exponential expression, the result is:

   A. 3^{9x}

   B. 27^x

   C. 3^{2x+1}

   D. 9^{3x}

   E. 9^{x^3}

---

6. In the figure below, the measure of angle A is:

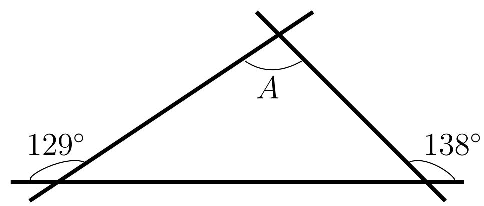
   A. $51°$

   B. $87°$

   C. $129°$

   D. $42°$

   E. $138°$



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7. Determine all points of intersection of the circle $(x-2)^2 + (y-1)^2 = 9$ and the line $y = 1$.

   A. $(1, 2)$

   B. $(0, 1)$ and $(4, 1)$

   C. $(-1, 1)$ and $(5, 1)$

   D. $(5, 1)$

   E. $(2, 1)$

---

8. The three solutions of the equation $f(x) = 0$ are -2, 0, and 3. Therefore, the three solutions of the equation $f(x-2) = 0$ are:

   A. -4, -2, and 1

   B. -2, 0, and 3

   C. 0, 2, and 5

   D. 2, 4, and 5

   E. None of these

---

9. The value of $\log_{49}(7) - \log_8(64)$ is:

A. $-\dfrac{2}{3}$

B. $-\dfrac{3}{2}$

C. $\dfrac{2}{5}$

D. $\dfrac{5}{2}$

E. None of these

---

10. It takes 6 cubes to build a staircase with 3 steps. The number of cubes needed to build a staircase with 11 steps is:

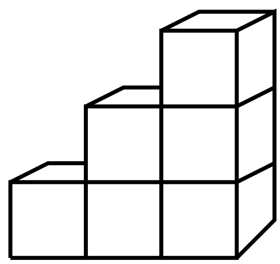
A. 96

B. 30

C. 78

D. 110

E. 66



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11. All of the solutions of $(x-3)^2 + (x-3) = -1 - (x-3)$ are contained in the interval:

A. $(-\infty, -2)$

B. $(-3, 0]$

C. $[-2, 1]$

D. $[0, 2)$

E. $[2, \infty)$

---

12. The number of positive integers less than 700 divisible by neither 5 nor 7 is:

A. 240

B. 220

C. 460

D. 480

E. None of these

---

13. When $\log_3 \sqrt{243\sqrt{81\sqrt[3]{3}}}$ is expressed as a rational number, the result is:

A. $\dfrac{2}{43}$

B. $\dfrac{43}{2}$

C. $\dfrac{43}{12}$

D. $\dfrac{43}{6}$

E. $\dfrac{43}{72}$

---

14. Point D is some interior point of ΔABC and x, the measure of $\angle ACB$ in degrees, is equal to:

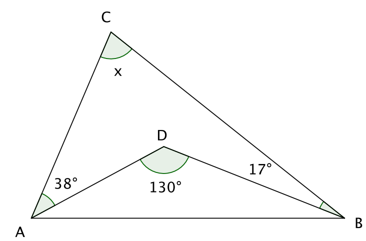
A. $50°$

B. $125°$

C. $92°$

D. $75°$

E. None of these



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15. For all x in the domain of the function $f(x) = \dfrac{x+1}{x^3 - x}$, an equivalent function is:

A. $f(x) = \frac{1}{x^2} - \frac{1}{x^3}$

B. $f(x) = \frac{1}{x^3} - \frac{1}{x}$

C. $f(x) = \frac{1}{x^2-1}$

D. $f(x) = \frac{1}{x^2-x}$

E. $f(x) = \frac{1}{x^3}$

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16. The sum of all of the integer solutions to $(x+4)(x-1)(x+2)^2 < 0$ is:

A. -9

B. -7

C. -2

D. -4

E. None of these

---

17. A **4x4 magic square** is made up of the numbers from 1 to 16 arranged in a 4x4 grid with a value S called the "magic sum." Magic squares have the following special properties: the sum of the entries in every row is S; the sum of the entries in every column is S; the sum of the entries along both diagonals is S. In the magic square below, the value of $2M + K$ is:

A. 34

B. 33

C. 23

D. 36

E. None of these

16	3	2	K
L	M	11	8
9	N	P	12
4	15	14	1

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18. The three squares have the dimensions indicated in the diagram. The area of the shaded trapezoid is:

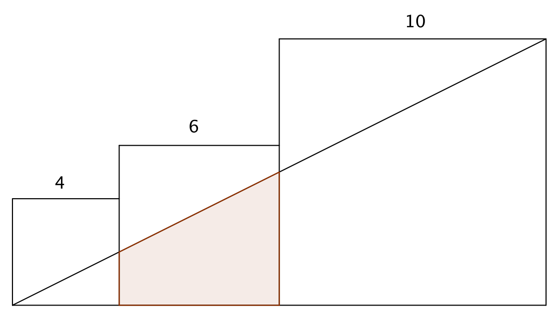
A. 21

B. 25

C. 42

D. 15

E. None of these



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19. A varsity soccer player kicks a soccer ball approximately straight upward with an initial velocity of 60 ft/sec. The formula for the height of the ball s (in feet) t seconds after being kicked is $s = -\frac{1}{2}gt^2 + v_0t + s_0$, where g is the acceleration due to gravity, v_0 is the initial velocity, and s_0 is the initial position of the ball. The ball leaves the player's foot at a height of 2 feet. Assume the acceleration due to gravity is 32 ft/sec^2. The time (or times) at which the ball is 52 feet in the air is:

A. The ball is 52 feet in the air after 1.25 seconds and 2.5 seconds.

B. The ball is 52 feet in the air after 1.5 seconds and 2.25 seconds.

C. The ball is 52 feet in the air only after 1.25 seconds.

D. The ball is 52 feet in the air only after 1.5 seconds.

E. None of these

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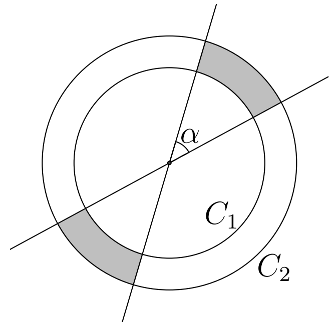
20. C_1 has radius $r_1 = 3$, C_2 has radius $r_2 = 4$, and $\alpha = 45°$. The area of the shaded region is:



A. $\frac{7\pi}{4}$

B. $\frac{7\pi}{2}$

C. 7π

D. $\frac{7\pi}{8}$

E. None of these

---

21. When all points (x, y) that satisfy $(x-3)^2 - (y-2)^2 = 0$ are graphed on the xy-plane, the result may best be described as:

A. A single point

B. A circle

C. A hyperbola

D. An ellipse

E. A pair of lines

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22. The values of θ for which $0 \le \theta \le \pi$ and $\sin^2\theta - \dfrac{3}{2} = -\dfrac{3}{2}\cos\theta$ are:

A. $\theta = 0, \dfrac{\pi}{4}$

B. $\theta = 0, \dfrac{\pi}{3}$

C. $\theta = 0, \dfrac{\pi}{2}$

D. $\theta = \dfrac{\pi}{3}, \dfrac{\pi}{4}$

E. $\theta = \dfrac{\pi}{3}, \dfrac{\pi}{2}$

---

23. Let x, y, and z be three numbers with $x < y < z$. The sum of the three numbers is 14. The largest is four times the smallest; the sum of the smallest and twice the largest is 18. The value of $z - 3y + 2x$ is:

A. -2

B. 0

C. 6

D. 14

E. None of these

---

24. Consider the equation $\sec^2(x) - 2\tan(x) = 4$. If k can be any integer, then which of the following is a subset of the set of solutions to the equation?

A. $x = \tan^{-1}(\sqrt{3}) + k\pi$

B. $x = \dfrac{\pi}{4} + 2k\pi$

C. $x = \tan^{-1}(3) + \dfrac{k\pi}{2}$

D. $x = \dfrac{3\pi}{4} + k\pi$

E. $x = \dfrac{\pi}{4} + k\pi$

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25. A race car driver completes a race by driving two laps around the same track. Her average speed for the entire two lap race was 96 miles per hour. The average speed for the first lap only was 80 miles per hour. The average speed for the second lap was:

A. 120 miles per hour

B. 88 miles per hour

C. 112 miles per hour

D. 110 miles per hour

E. None of these

---

26. For real numbers a and b, define

$$a \circ b = a + b + ab.$$

The value of y for which $x \circ 2 = -5 \circ y$ is:

A. $x \circ 7$

B. $-\frac{1}{5}x \circ -\frac{2}{5}$

C. $\frac{1}{4}(3x + 7)$

D. $-\frac{1}{4}(3x - 7)$

E. None of these

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27. The perimeter of a square that is inscribed in a circle with diameter 8 inches is:

A. $4\sqrt{2}$

B. $2\sqrt{2}$

C. $16\sqrt{2}$

D. $8\sqrt{2}$

E. None of these

---

28. In the equation $x^2 + mx + n = 0$, the values m and n are integers. The ONLY possible value for x is -3. The value of m is:

A. -6
B. -9
C. 3
D. 6
E. 9

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29. Shortly after a heist at the Museum of Math, a group of mendacious mathematicians were apprehended and interviewed. The statements below were given to the interrogator. If each mathematician told exactly one lie, the person who committed the heist was:

A. Alfred
B. Brian
C. Charles
D. Dan
E. Ed

Alfred:	It was not Brian; it was Dan.
Brian:	It was Alfred; it was not Ed.
Charles:	It was Brian; it was not Dan.
Dan:	It was not Charles; it was not Ed.
Ed:	It was Brian; it was Charles.

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30. The expression $\dfrac{\sin^4 x - \cos^4 x}{\alpha\left(\sin^2 x - \cos^2 x\right)}$ is equivalent to:

A. $\dfrac{\alpha}{4}$
B. $\dfrac{\alpha}{2}$
C. α
D. $\dfrac{1}{\alpha}$
E. $\dfrac{2}{\alpha}$

---

31. A motorist who averages 60 miles per hour leaves a town at 1 PM. A half hour later, another motorist, averaging 10 miles per hour slower leaves a destination 250 miles from the first town to meet the other motorist. Assuming both motorists maintain their average speeds, the time they meet is:

A. 2:30 PM
B. 3:00 PM
C. 3:30 PM
D. 4:00 PM
E. 4:15 PM

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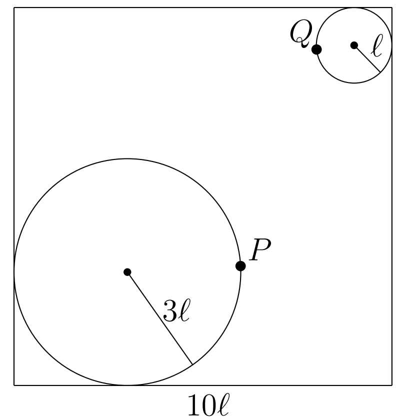
32. The enclosing square has sides of length 10ℓ. By adjusting the positions of points P and Q along each respective circle, the shortest distance between the points is:



A. 2ℓ

B. 4ℓ

C. $\left(6\sqrt{2} - \frac{9}{2}\right)\ell$

D. $(6\sqrt{2} - 4)\ell$

E. None of these

---

33. Consider the equation

$$2\log_b(x) = 2\log_b(1-a^2) - \log_b\left(\frac{1}{a} - a\right)^2 .$$

Assuming a and b are positive real numbers, a solution to the equation is:

A. $-a$

B. a

C. b^a

D. a^b

E. The equation has no solution

---

34. A *rook* is a chess piece that moves either horizontally or vertically on a chessboard. Two rooks, say R1 and R2, can attack each other provided that R1 and R2 are either both on the same row or both on the same column of a chessboard; and, no other piece is between the two rooks R1 and R2. A standard chessboard comprises 8 rows and 8 columns, thus having a total of 64 squares. The number of ways that five rooks, R1, R2, R3, R4, and R5, can be placed on a standard 8×8 chessboard such that no two of the five rooks are attacking each other is equal to:

A. 3136

B. 56

C. 120

D. 240

E. None of these

---

35. The solution set to $\dfrac{17x}{x^2-5x+6} - \dfrac{6}{x^2-x-2} \geq \dfrac{8x}{x^2-2x-3}$ is:

A. $(-\infty,-2)\cup(3,\infty)$
B. $[-2,-1)\cup(-1,2)\cup(3,\infty)$
C. $(-1,2)\cup(3,\infty)$
D. $(-\infty,-2)\cup(2,3)$
E. None of these

---

36. Two consecutive even integers that are both solutions of the equation $\sin^{-1}\left(\sin\left(\dfrac{x\pi}{6}\right)\right) = -\dfrac{\pi}{3}$ and whose product P satisfies $200 < P < 300$ are:

A. 14, 16
B. -14, -16
C. 16, 18
D. -16, -18
E. None of these

---

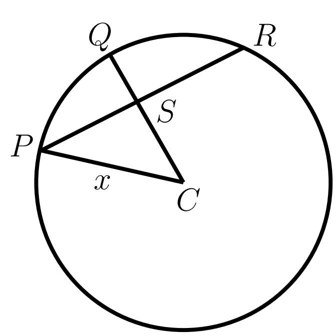
37. The set of real solutions to $(x^2-2x-3)(x^2+3x-4)-(x^2+x-12)=0$ is:

A. $\{-4,-1,1,3\}$
B. $\{-4,-\sqrt{2},\sqrt{2},3\}$
C. $\{-4,1,3\}$
D. $\{-4,-3,-1,1\}$
E. None of these

---

38. In the circle below with center C, we know $\overline{CQ}\perp\overline{PR}$, $\overline{PR}$ is 12 units long, and $\overline{SQ}$ is 2 units long. The value of x is:

A. 10
B. 6
C. 8
D. 14
E. None of these



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39. A permutation of the numbers 1, 2, ..., n is a function $f(x)$ from the domain $X = \{1, 2, \ldots, n\}$ to the range $Y = \{1, 2, \ldots, n\}$ such that for each $y \in Y$, there exists an $x \in X$ with $f(x) = y$. A permutation can be represented by the diagram

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}.$$

For instance, one possible permutation of the numbers 1, 2, 3, 4 is represented by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix},$$

which represents the function $f(x)$ satisfying $f(1) = 4$, $f(2) = 1$, $f(3) = 2$, and $f(4) = 3$.

An **inversion** of a permutation $f(x)$ is a pair of distinct domain values x_1, x_2 for which $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. The number of distinct inversions in the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 5 & 1 & 2 & 8 & 3 & 6 & 4 \end{pmatrix}$$

is equal to:

A. 4

B. 7

C. 14

D. 28

E. None of these

---

40. The graph of a cubic polynomial $y = ax^3 + bx^2 + cx + d$ is as shown. The sum of the coefficients a, b, c and d is:

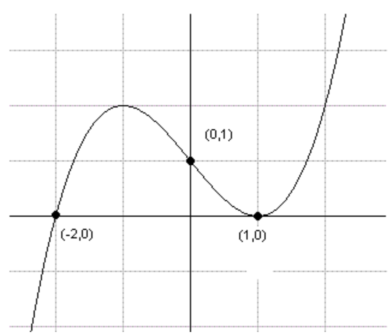
A. 0

B. 3

C. -1

D. 1.5

E. None of these



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41. The number of different x values that are solutions to $|x^2 + 4x + 3| - |x^2 - 2x - 3| = 5$ is:

A. 0

B. 1

C. 2

D. 3

E. 4

---

42. Suppose $4^{2x+y} = 256$ and $4^{x-y} = \frac{1}{16}$. The value of y is:

A. $\dfrac{2}{3}$

B. $\dfrac{3}{2}$

C. $\dfrac{8}{3}$

D. $\dfrac{2}{5}$

E. $\dfrac{3}{5}$

---

43. In the figure below. the smaller circle is tangent to the larger circle and is also tangent to the sides of the square. The length of r_2 is:

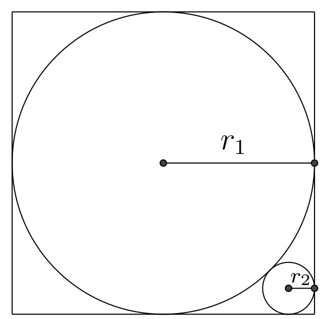
A. $(\pi - 3)r_1$

B. $(2\sqrt{2} - 1)r_1$

C. $(2\sqrt{3} - 1)r_1$

D. $(4 - 2\sqrt{2})r_1$

E. None of these



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44. Given that the terms $\log(2), \log(2^x - 1),$ and $\log(2^x + 3)$ are in an arithmetic sequence, the value of x is:

A. $\dfrac{3}{2}$

B. $\dfrac{2}{3}$

C. $\log_2(3)$

D. $\log_2(5)$

E. $\log_5(2)$

---

45. Solve $2x^2 + 4x = 1$. Label the roots as a and b, where $a < b$. Calculate $a \div b$

A. $-1 - \frac{\sqrt{6}}{2}$

B. $-2 + \sqrt{3}$

C. -1

D. $-1 \pm \frac{\sqrt{6}}{2}$

E. $-5 - 2\sqrt{6}$

---

46. The set A is a subset of the set B provided that every member of A is also a member of B. By default, the empty set, $\emptyset$, is a subset of every set. Suppose that $X = \{a,\, b,\, x_1,\, x_2,\, \ldots,\, x_{n-2}\}$ is a set containing exactly n distinct objects. The number of subsets of X that do not contain both a and b is equal to:

A. 6^{n-2}

B. $3 \cdot 2^{n-2}$

C. 2^{n-2}

D. $n - 2$

E. None of these

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47. Solutions of $2(5^{x+1}) = 1 + \dfrac{3}{5^x}$ can be expressed in the form $x = m + \dfrac{\log(n)}{\log(5)}$. One possible product of m and n is:

A. -3

B. 0

C. -1

D. 2

E. 1

---

48. Consider the following system of equations.

$$\begin{aligned} x^2 + y^2 &= 25 \\ 4x - 2y &= zx \\ -2x + y &= zy \end{aligned}$$

The number of ordered triples (x, y, z) that are solutions to this system and that have all strictly positive coordinates is:

A. 0

B. 1

C. 2

D. 3

E. 4

---

49. The sequence $\{C_0, C_1, C_2, \ldots\}$ is called the sequence of *Catalan numbers*. For $n > 0$, the Catalan number C_n may be computed by the equation

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k} = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0 \ .$$

Starting with $C_0 = 1$ and $C_1 = 1$, we can compute that the Catalan number C_5 is:

A. 20

B. 32

C. 40

D. 42

E. None of these

---

50. In 1697, Johann Bernoulli solved the Brachistochrone problem: If a marble is placed at P, what is the curve C that minimizes the time of descent to point Q? (We are assuming only gravity and no friction.) He found C to be that of a **cycloid**. A cycloid is generated by tracing a fixed point, initially located at P, as the circle is rolled along line l. The length of C is:

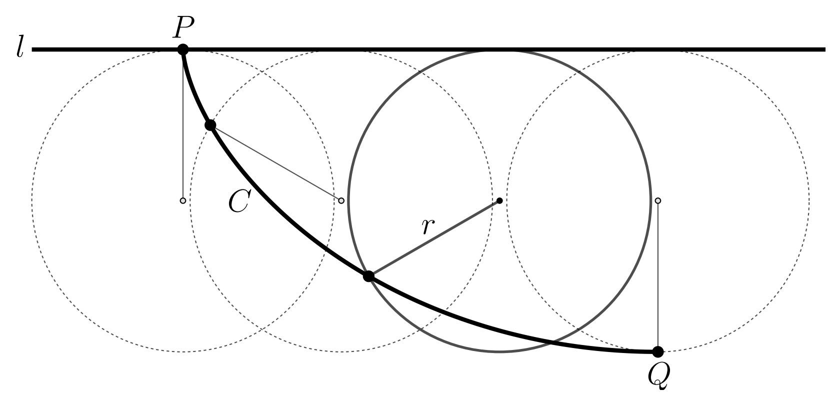
A. $2r$

B. $r\sqrt{4+\pi}$

C. $r\sqrt{4+\pi^2}$

D. $4r$

E. $(2+\pi)r$



Answer Key

1. A
2. C
3. C
4. E
5. C
6. B
7. C
8. C
9. B
10. E
11. E
12. D
13. C
14. D
15. D
16. D
17. B
18. A
19. A
20. A
21. E
22. B
23. B
24. D
25. A
26. E
27. C
28. D
29. C
30. D
31. C
32. D
33. B
34. A
35. B
36. B
37. B
38. A
39. C
40. A
41. B
42. C
43. E
44. D
45. E
46. B
47. A
48. A
49. D
50. D